REMARKS ON THE RESOLUTION IN INTEGERS OF THE EQUATION $X(X+1)(X+2)\ldots(X+N)=Y^2$ (1), *AND SOLUTION OF PROB.* 262.

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EVERY divisor common to any two of the integral numbers x, $(x+1)$, $(x+2)$, ... $(x+n)$, since it must divide their difference, is necessarily one of the numbers 1, 2, 3, 4, ... n. Therefore every prime number that is a divisor of any two of the integral numbers x, $(x+1)$, $(x+2)$, ... $(x+n)$ is necessarily one of the prime numbers 2, 3, 5, ... p, where p is the greatest of the prime numbers comprised in the series 1, 2, 3, 4, ... n.

If the product $x(x+1)(x+2)\ldots(x+n)$ is an exact square, as equation (1) supposes, and if the exponent of the highest power of a prime number dividing one of its factors is odd, that prime number will belong to the series 2, 3, 5, ... p; for it will divide at least two of the factors of the product: hence we conclude that:

1°. *If a factor of the product* $x(x+1)(x+2)\ldots(x+n)$ *is not divisible by any of the numbers* 2, 3, 5,...p, *it will be, for this reason, an exact square.*

2°. *No one of the factors* x, $(x+1)$, $(x+2)$, ... $(x+n)$ *can admit as divisors all the numbers* 2, 3, 5, ... p.

We now proceed to demonstrate 2°. Let us in the first place suppose that one of the two extreme factors, for example x, is separately divisible by $2.3.5\ldots p$. Then $(x+1)$, which is prime to x, not admitting any one of these divisors, will be an exact square (1°); and as x is, by hypothesis, a multiple of $(2.3.5\ldots p)$, the factors $(x+2)$, $(x+4)$ will be divisible by 2 and will not have another divisor comprised in the series 2, 3, 5, ... p. Moreover, neither of the factors $(x+2)$, $(x+4)$, is an exact square, for $(x+1)$ being the square of an integer greater than unity, no other number greater than $x+1$ can be the square of a whole number unless it exceeds $x+1$ by at least two units. It is necessary then (1°) that $(x+2)$, $(x+4)$, should be whole numbers of the form $2a^2$, $2b^2$, and this requires that $b^2-a^2=1$. We have therefore arrived at this conclusion, that if x is a multiple of $(2.3.5\ldots p)$, the difference of the squares of two whole numbers will be unity.

The same demonstration applies to the other extreme factor $(x+n)$.

With reference to the intermediate factors $(x+1)$, $(x+2)$, ... $(x+n-1)$, if one of them, $(x+1)$ for example, were a multiple of $(2.3.5\ldots p)$, the two factors x and $(x+2)$ which include it, would be squares (1°); which is evidently impossible, since they differ by only two units.

Applications of the Foregoing.

[If $x(x+1)\ (x+2)\ (x+3)\ (x+4) = y^2$ (2), then $p = 3$, $(2\,.\,p) = 6$. and (2°), $x = 6m+1$; and (1°), $x = 6m+1 = a^2$ and $x+4 = 6m+5 = b^2$; whence $b^2—a^2 = 4$; which is evidently impossible in integers. Translator.]

[*Solution of Problem IV., Mathematical Monthly, Vol. I., No. II.*]—If $x(x+1)(x+2)(x+3)(x+4)(x+5) = y^2$ (3), $p = 5$, $(2.3\,.\,p) = 30$.

We will proceed to show that if (3) admits of a solution in integers, it produces the absurdity that, of six consecutive integers, no one is divisible by 6.

If $x = 6a$, the whole number a is not divisible by 5 (2°). Now the equation $x = 6a$ gives $x+5 = 6a+5$: moreover we see that $x+5$ is prime to 2, 3, 5; then $(x+5)$ is a square (1°). It follows that $x(x+1)(x+2)(x+3) \times(x+4)$, the product of five consecutive integers, would be a square; that which never takes place.

The same demonstration applies to the other extreme factor $(x+5)$.

It remains to consider the remaining intermediate factors $(x+1)$, $(x+2)$, $(x+3)$, $(x+4)$. If one of them, $(x+2)$ for example, is a multiple of 6 it is evident that the two factors $(x+3)$, $(x+1)$ which comprise $(x+2)$, will be prime to 6; and as they cannot be both multiples of 5, one of them will not admit of any of the divisors 2, 3, 5, and will therefore be the square, a^2, of a whole number.

The other factor, $a^2 \pm 2$, not being a square ought to be divisible by 5 (1°). We shall then have $a^2 \pm 2 = 5m$; now this equation cannot exist, for we know that the formula for squares that are not multiples of 5, is $a^2 = 5m \pm 1$. It follows from what precedes that the product of six consecutive integers cannot be a square.

[*Solution of Problem* 262, *Analyst, Vol. VI., No.* 3.]—If $x(x+1)\ (x+2) \times(x+3)\ (x+4)\ (x+5)\ (x+6) = y^2$ (4), admitted of a solution in integers, no one of the seven numbers x, $(x+1)$, $(x+2)$, $(x+3)$, $(x+4)$, $(x+5)$, $(x+6)$ would be divisible by 6. In reality p being still equal to 5, we will demonstrate as in the preceding case, that it is impossible that one of the factors $(x+1)$, $(x+2)$, $(x+3)$, $(x+4)$, $(x+5)$ can be a multiple of 6.

We will now demonstrate that the extreme factors x, $(x+6)$ cannot be multiples of 6. Supposing $x = 6m$, we shall have $x+6 = 6(m+1)$; and neither of the members m, $(m+1)$ will be divisible by 5 (2°). Now, the equalities $x = 6m$, $x+6 = 6(m+1)$ give $x+1 = 6(m+1)—5$ and $x+5 = 6m +5$. These last show that the factors $(x+1)$, $(x+5)$ are not divisible by any of the three prime factors 2, 3, 5. Then (1°) we must have $x+1 = a^2$, $x+5 = b^2$, whence $b^2—a^2 = 4$; an equality which is impossible, since it requires that the difference of the squares of two integers b, a, greater than unity, be less than 5. It is also proved that the proposed equation (4) does not admit of a solution in integers.